A Comparison of Products in Hochschild Cohomology

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Abstract. We compare Gerstenhaber's cup product on the Hochschild cochain complex with the topological (simplicial) cup product on the hom-dual of the b-complex from Hochschild homology. When $A = k[G]$ is a group ring, there are cochain maps between these two cochain complexes, which allows a precise study of products. Considering the classifying space BG as a natural subspace of $\mathrm{Maps}(S^1,\ BG)$, we prove that for cochains supported on BG, the simplicial cup product agrees with Gerstenhaber's product, and Steenrod's cup-one product agrees with Gerstenhaber's pre-Lie product (up to cochain isomorphism). Moreover, the entire family of Steenrod's cup-i products are defined on the hom-dual of the b-complex, defining an action of the Steenrod algebra on Hochschild cohomology. These results are extended to simplicial group rings $k[G_*]$. When G_* is a simplicial group model for the based loop space on a connected CW-complex X, we recover the action of the mod-2 Steenrod algebra on the singular cohomology of the free loop space via Hochschild hypercohomology.

Key Words: Hochschild cohomology, Gerstenhaber's product, cup-i products, simplicial groups, the free loop space.

1 Introduction

The results of the paper are motivated via the toy model provided by the Hochschild cohomology groups $HH^*(A;\ A)$ when A is the group ring $\mathbf{Z}/2[\mathbf{Z}/2]$. Two products are supported on these cohomology groups. The first is Gerstenhaber's cup product [5], defined on Hochschild's original cochain complex [7] for computing $HH^*(A;\ A)$, and involves the product in A. The second is the simplicial (topological) cup product on the Hom-dual of the b-complex

defining Hochschild homology, and involves the product in the ground ring k. When $A = \mathbf{Z}/2[\mathbf{Z}/2]$, these two operations can be interpreted as products on the singular cohomology groups (with $\mathbf{Z}/2$-coefficients) of the space

$$\mathrm{Maps}(S^1,\ B\mathbf{Z}/2) \simeq \mathbf{R}P^\infty \sqcup \mathbf{R}P^\infty,$$

one copy of $\mathbf{R}P^\infty$ corresponding to contractible maps of S^1 into $B\mathbf{Z}/2$, and the other copy corresponding to essential (non-contractible) maps of S^1 into $B\mathbf{Z}/2$. It is easily seen that on the component of $\mathrm{Maps}(S^1,\ B\mathbf{Z}/2)$ corresponding to the contractible maps, the Gerstenhaber and simplicial cup products agree. Moreover, Gerstenhaber's pre-Lie product agrees with Steenrod's cup-one product on this component. Thus, Steenrod's cup-2 product is a measure of Gerstenhaber's pre-Lie product to be non-commutative on cochains of this component (up to cochain isomorphism). In this way, Steenrod's cup-i products can be interpreted as higher measures of the non-commutativity of Gerstenhaber's cup product on cochains. The purpose of this paper is to generalize these results to group rings and simplicial group rings, such as models for the based loop space of a connected CW-complex. We work with integer coefficients, or more generally with coefficients in an associative, commutative, unital ground ring k.

For any group ring $k[G]$, there is a natural map of cochain complexes

$$\Phi_n : \mathrm{Hom}_k(k[G]^{\otimes n},\ k[G]) \to \mathrm{Hom}_k(k[G]^{\otimes(n+1)},\ k), \quad n \geq 0,$$

induced by an inner product $\langle\ ,\ \rangle : k[G] \otimes k[G] \to k$ arising from the structure of $k[G]$ as an algebra over the cyclic operad [10, 13.14.6]. Gerstenhaber's cup product and pre-Lie product are defined on the cochain complex

$$\mathrm{Hom}_k(k[G]^{\otimes *},\ k[G]),$$

while the simplicial (topological) cup product and Steenrod's cup-i products are defined on the cochain complex

$$\mathrm{Hom}_k(k[G]^{\otimes(*+1)},\ k).$$

The latter uses the face maps of the cyclic bar construction $N^{\mathrm{cy}}_*(G)$. Letting $B_*(G)$ denote the simplicial bar construction on G, there are maps of simplicial objects:

$$\iota : B_*(G) \to N^{\mathrm{cy}}_*(G), \quad \pi : N^{\mathrm{cy}}_*(G) \to B_*(G)$$

such that $\pi \circ \iota = \mathbf{1}$ on $B_*(G)$. This allows the construction of cochains in $\mathrm{Hom}_k(k[G]^{\otimes(*+1)},\ k)$ that are supported on BG. Moreover, there is a simple description of a cochain map

$$\Psi_n : \mathrm{Hom}_k(k[G]^{\otimes(n+1)},\ k) \to \mathrm{Hom}_k(k[G]^{\otimes n},\ k[G])$$

that is in fact an isomorphism of cochain complexes when G is finite. We prove that for cochains supported on BG, Ψ maps the simplicial cup product to Gerstenhaber's cup product, with equality as cochains. Moreover, for such cochains, Ψ maps Steenrod's cup-one product to Gerstenhaber's pre-Lie product, with equality as cochains. Using the simplicial face maps of $N_*^{\mathrm{cy}}(G)$, the entire family of Steenrod's cup-i products are defined on $\mathrm{Hom}_k(k[G]^{\otimes(*+1)},\ k)$. For cochains supported on BG, these map to higher measures of Gerstenhaber's cup product to be non-commutative at the level of cochains.

Now let G_* be a simplicial group, such as a model for ΩX, the based loop space of a connected CW-complex X. By work of Goodwillie [6] and Burghelea and Fiedorowicz [3], we have

$$||N_*^{\mathrm{cy}}(G_*)|| \simeq \mathrm{Maps}(S^1,\ X) \quad \text{and} \quad ||B_*(G_*)|| \simeq X.$$

Considering $N_*^{\mathrm{cy}}(G_*)$ as a bisimplicial object, we form a certain bicomplex with columns given by $(N_*^{\mathrm{cy}}(k[G_p]),\ b)$ and rows given by $((k[G_*])^{\otimes(n+1)},\ d)$, whose total homology is $HH_*(k[G_*];\, k[G_*])$. The Hom_k dual of this bicomplex, called the $(b^*,\, d^*)$-complex (3.3), has cohomology isomorphic to the singular cohomology groups $H^*(\mathcal{L}X;\, k)$, where $\mathcal{L}(X) = \mathrm{Maps}(S^1,\ X)$. Moreover, a bisimplicial cup product can be defined on the $(b^*,\, d^*)$-complex, and there are bisimplicial versions of Steenrod's cup-i product. We explicitly construct the cup-one product on $(b^*,\, d^*)$ cochains. This induces an action of the Steenrod algebra on the mod-2 cohomology of the $(b^*,\, d^*)$-complex. The homotopy equivalence

$$||N_*^{\mathrm{cy}}(G_*)|| \simeq \mathcal{L}(X)$$

induces an isomorphism of modules over the Steenrod algebra from the singular cohomology $H^*(\mathcal{L}(X);\, \mathbf{Z}/2)$ to the cohomology of the $(b^*,\, d^*)$-complex, i.e., $HH^*(\mathbf{Z}/2[G_*];\, \mathbf{Z}/2[G_*])$.

Finally, we construct a hybrid cochain bicomplex with columns

$$(\mathrm{Hom}_k(k[G_p]^{\otimes *},\ k[G_p]),\ \delta)$$

and rows $(\mathrm{Hom}_k(k[G_*]^{\otimes n},\ k[G_*]),\ D)$, where D is a certain sum of inverse images of the individual face operators $d_i : G_{p+1} \to G_p$. We call this the $(\delta,\ D)$-cochain complex. The cochain map Φ_n above extends to a map of cochain complexes, from the $(\delta,\ D)$-complex to the $(b^*,\ d^*)$-complex. When each G_p is finite, Φ_n is an isomorphism of cochain complexes.

2 Hochschild Cohomology

Let A be an associative algebra over a ground ring k that is unital, commutative and associative. Recall [7, 8] that $HH^*(A;\,A)$, the Hochschild cohomology of A with coefficients in A viewed as a bimodule over itself is the homology of the cochain complex:

$$\mathrm{Hom}_k(k,\ A) \xrightarrow{\delta} \mathrm{Hom}_k(A,\ A) \xrightarrow{\delta} \dots$$
$$\dots \xrightarrow{\delta} \mathrm{Hom}_k(A^{\otimes n},\ A) \xrightarrow{\delta} \mathrm{Hom}_k(A^{\otimes(n+1)},\ A) \xrightarrow{\delta} \dots,$$

where, for a k-linear map $f : A^{\otimes n} \to A$, $\delta f : A^{\otimes(n+1)} \to A$ is given by

$$(\delta f)(a_1,\ a_2,\ \dots,a_{n+1}) = a_1 f(a_2,\ \dots,a_{n+1}) +$$
$$\sum_{i=1}^{n}(-1)^i f(a_1,\ a_2,\ \dots,a_i a_{i+1},\ \dots,a_n) + (-1)^{n+1} f(a_1,\ a_2,\ \dots,a_n)a_{n+1}.$$

For the special case of $n = 0$, $(\delta f)(a_1) = a_1 f(1) - f(1)a_1$.

These cohomology groups, $HH^*(A;\,A)$, support a construction as an EXT-functor over the ring $A \otimes A^{\mathrm{op}}$ [4, IX.4]. Also, applying the TOR-functor to a certain free $A \otimes A^{\mathrm{op}}$ resolution of the product

$$m : A \otimes A \to A, \quad m(x \otimes y) = xy,$$

we have the following standard resolution for computing $HH_*(A;\,A)$, the Hochschild homology of A with coefficients in the bimodule A [12, X.4]:

$$A \xleftarrow{b} A^{\otimes 2} \xleftarrow{b} \dots \xleftarrow{b} A^{\otimes n} \xleftarrow{b} A^{\otimes(n+1)} \xleftarrow{b} \dots,$$

where for $(a_0,\ a_1,\ \dots,a_n) \in A^{\otimes(n+1)}$,

$$b(a_0,\ a_1,\ \dots,a_n) =$$
$$\Big(\sum_{i=0}^{n-1}(-1)^i(a_0,\ \dots,a_i a_{i+1},\ \dots,a_n)\Big) + (-1)^n(a_n a_0,,a_1,\ \dots,a_n).$$

For $n = 1$, $b(a_0,\, a_1) = a_0a_1 - a_1a_0$. Moreover, when A is unital, $\{A^{\otimes(n+1)}\}_{n\geq 0}$ is a simplicial k-module with face maps

$$d_i = b_i : A^{\otimes(n+1)} \to A^{\otimes n}, \quad i = 0,\ 1,\ 2,\ \ldots,\ n, \tag{2.1}$$
$$b_i(a_0,\, a_1,\, \ldots,\, a_n) = (a_0,\, \ldots,\, a_ia_{i+1},\, \ldots,\, a_n), \quad 0 \leq i \leq n-1, \tag{2.2}$$
$$b_n(a_0,\, a_1,\, \ldots,\, a_n) = (a_na_0,\, ,a_1,\, \ldots,\, a_n), \tag{2.3}$$

and degeneracies $s_i : A^{\otimes(n+1)} \to A^{\otimes(n+2)}$, $i = 0,\, 1,\, 2,\, \ldots,\, n$,

$$s_i(a_0,\, a_1,\, \ldots,\, a_n) = (a_0,\, a_1,\, \ldots,\, a_i,\, 1,\, a_{i+1},\, \ldots,\, a_n), \quad 0 \leq i \leq n. \tag{2.4}$$

Let $HH^*_{\mathcal{K}}(A)$ denote the homology of the cochain complex $\mathrm{Hom}_k(A^{\otimes(*+1)},\, k)$, i.e.,

$$\mathrm{Hom}_k(A,\, k) \xrightarrow{b^*} \mathrm{Hom}_k(A^{\otimes 2},\, k) \xrightarrow{b^*} \ldots$$
$$\ldots \xrightarrow{b^*} \mathrm{Hom}_k(A^{\otimes n},\, k) \xrightarrow{b^*} \mathrm{Hom}_k(A^{\otimes(n+1)},\, k) \xrightarrow{b^*} \ldots,$$

where for a k-linear map $\varphi : A^{\otimes n} \to k$, $b^*(\varphi) : A^{\otimes(n+1)} \to k$ is given by

$$b^*(\varphi)(a_0,\, \ldots,\, a_n) = \varphi(b(a_0,\, \ldots,\, a_n)).$$

Both $HH^*(A;\, A)$ and $HH^*_{\mathcal{K}}(A)$ inherit graded product structures from associative products on the respective cochains. For $f \in \mathrm{Hom}_k(A^{\otimes p},\, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes q},\, A)$, the Gerstenhaber (cup) product [5]

$$f \underset{G}{\cdot} g \in \mathrm{Hom}_k(A^{\otimes(p+q)},\, A)$$

is given by

$$(f \underset{G}{\cdot} g)(a_1,\, a_2,\, \ldots,\, a_{p+q}) = f(a_1,\, \ldots,\, a_p) \cdot g(a_{p+1},\, \ldots,\, a_{p+q}),$$

where the product above occurs in the algebra A. Then

$$\delta(f \underset{G}{\cdot} g) = (\delta f) \underset{G}{\cdot} g + (-1)^p f \underset{G}{\cdot} (\delta g).$$

Thus, if $f \in HH^p(A;\, A)$ and $g \in HH^q(A;\, A)$, then $f \underset{G}{\cdot} g \in HH^{p+q}(A;\, A)$. For $\alpha \in \mathrm{Hom}_k(A^{\otimes(p+1)},\, k)$ and $\beta \in \mathrm{Hom}_k(A^{\otimes(q+1)},\, k)$, the simplicial (cup) product $\alpha \underset{S}{\cdot} \beta \in \mathrm{Hom}_k(A^{\otimes(p+q+1)},\, k)$ is given by

$$(\alpha \underset{S}{\cdot} \beta)(\sigma) = \alpha(d_{p+1} \circ d_{p+2} \circ \ldots \circ d_{p+q}(\sigma)) \cdot \beta(d_0 \circ d_1 \circ \ldots \circ d_{p-1}(\sigma)),$$

where $d_{p+1} \circ d_{p+2} \circ \ldots \circ d_{p+q}(\sigma)$ is the front p-face of $\sigma = (a_0, a_1, \ldots, a_{p+q}) \in A^{\otimes p+q+1}$ and $d_0 \circ d_1 \circ \ldots \circ d_{p-1}(\sigma) = d_0^p(\sigma)$ is the back q-face of σ. The product above is now in the ground ring k. We have

$$b^*(\alpha \underset{S}{\cdot} \beta) = b^*(\alpha) \underset{S}{\cdot} \beta + (-1)^p \alpha \underset{S}{\cdot} b^*(\beta).$$

For $\alpha \in HH^p_{\mathcal{K}}(A)$ and $\beta \in HH^q_{\mathcal{K}}(A)$, it follows that $\alpha \underset{S}{\cdot} \beta \in HH^{p+q}_{\mathcal{K}}(A)$.

Gerstenhaber [5] has shown that on $HH^*(A;\, A)$, the product $f \underset{G}{\cdot} g$ is graded commutative by using the idea of function composition, understood today in terms of the endomorphism operad $\mathrm{Hom}_k(A^{\otimes n},\, A)$ [10, 5.2.12]. Specifically, for $f \in \mathrm{Hom}_k(A^{\otimes p},\, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes q},\, A)$, define $f \underset{(j)}{\circ} g \in \mathrm{Hom}_k(A^{\otimes p+q-1},\, A)$ for $j = 0, 1, 2, \ldots, p-1$, by

$$(f \underset{(j)}{\circ} g)(a_1, a_2, \ldots, a_{p+q-1}) =$$
$$f(a_1, \ldots, a_j, g(a_{j+1}, \ldots, a_{j+q}), a_{j+q+1}, \ldots, a_{p+q-1}).$$

Choosing the sign convention $f \circ g = \sum_{j=0}^{p-1} (-1)^{(p-1-j)(q-1)} f \underset{(j)}{\circ} g$, we have

$$\delta(f \circ g) = (\delta f) \circ g + (-1)^{p-1}(f \circ \delta g) + (-1)^p [f \underset{G}{\cdot} g - (-1)^{pq} g \underset{G}{\cdot} f].$$

If f and g are cocycles, then $f \underset{G}{\cdot} g$ and $(-1)^{pq} g \underset{G}{\cdot} f$ differ by a coboundary, so that in $HH^*(A;\, A)$,

$$f \underset{G}{\cdot} g = (-1)^{pq} g \underset{G}{\cdot} f.$$

Gerstenhaber calls $f \circ g$ a pre-Lie product, since $f \circ g - (-1)^{(p+1)(q+1)} g \circ f$ induces a Lie bracket on $HH^*(A;\, A)$.

From the work of Steenrod [15], it follows that the simplicial cup product is graded commutative on the cohomology of any simplicial complex, although in 1947 Steenrod was writing just before the advent of (semi) simplicial complexes. For $\alpha \in \mathrm{Hom}_k(A^{\otimes(p+1)},\, k)$ and $\beta \in \mathrm{Hom}_k(A^{\otimes(q+1)},\, k)$, recall that the cup-one product

$$\alpha \underset{1,S}{\cdot} \beta \in \mathrm{Hom}_k(A^{\otimes(p+q)},\, k)$$

can be written in terms of the face maps d_i as

$$(\alpha \underset{1,S}{\cdot} \beta)(\sigma) = \sum_{j=0}^{p-1} (-1)^{(p-1-j)(q-1)}\, \alpha((d_{j+1} \circ d_{j+2} \circ \ldots \circ d_{j+q-1})(\sigma)) \cdot \beta((d_0 \circ d_1 \circ \ldots \circ d_{j-1} \circ d_{j+q+1} \circ d_{j+q+2} \circ \ldots \circ d_{p+q-1})(\sigma)),$$

where $\sigma = (a_0, a_1, \ldots, a_{p+q-1}) \in A^{\otimes p+q}$. With the above choice of signs, we have:

$$b^*(\alpha \underset{1,S}{\cdot} \beta) = b^*(\alpha) \underset{1,S}{\cdot} \beta + (-1)^{p-1} \alpha \underset{1,S}{\cdot} b^*(\beta) + (-1)^p [\alpha \underset{S}{\cdot} \beta - (-1)^{pq} \beta \underset{S}{\cdot} \alpha].$$

Again, for cocycles α and β, $\alpha \underset{S}{\cdot} \beta$ and $(-1)^{pq}\beta \underset{S}{\cdot} \alpha$ differ by a coboundary. Today the cup-one products and their higher analogues (cup-i, $i \geq 1$) are understood in terms of operads [1, 11].

Let G be a discrete group and let $B_*(G)$ be the simplicial bar construction on G. By definition, $B_n(G) = G^n$, $n = 0, 1, 2, \ldots$, with face maps

$$d_i : G^n \to G^{n-1}, \quad i = 0, 1, 2, \ldots, n,$$
$$d_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n), & i = 0, \\ (g_1, g_2, \ldots, g_i g_{i+1}, \ldots, g_n), & i = 1, 2, \ldots, n-1, \\ (g_1, g_2, \ldots, g_{n-1}), & i = n, \end{cases}$$

and degeneracies

$$s_i : G^n \to G^{n+1}, \quad i = 0, 1, \ldots, n,$$
$$s_i(g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n).$$

Of course, the geometric realization $|B_*(G)|$ is a model for the classifying space BG, up to homotopy. Let $N_*^{\mathrm{cy}}(G)$ denote the cyclic bar construction [9, 7.3.10] on G with $N_n^{\mathrm{cy}}(G) = G^{n+1}$, $n = 0, 1, 2, \ldots$. The face maps $d_i : G^{n+1} \to G^n$ and degeneracies $s_i : G^{n+1} \to G^{n+2}$ are given by adopting the formulas (2.1)–(2.4). For the geometric realization, we have [9, 7.3.11]

$$|N_*^{\mathrm{cy}}(G)| \simeq \mathrm{Maps}(S^1, BG) := \mathcal{L}BG,$$

where S^1 denotes the unit circle. Thus, $HH_*(k[G]; k[G]) \simeq H_*(\mathcal{L}BG; k)$.

There are maps of simplicial sets

$$\begin{aligned}
&\iota : B_*(G) \to N^{\mathrm{cy}}_*(G), \quad \pi : N^{\mathrm{cy}}_*(G) \to B_*(G),\\
&\iota : G^n \to G^{n+1}, \quad \pi : G^{n+1} \to G^n\\
&\iota(g_1,\, g_2,\, \dots\,,\, g_n) = ((g_1 g_2 \dots g_n)^{-1},\, g_1,\, g_2,\, \dots\,,\, g_n)\\
&\pi(g_0,\, g_1,\, g_2,\, \dots\,,\, g_n) = (g_1,\, g_2,\, \dots\,,\, g_n)
\end{aligned}$$

Let $\iota_* : H_*(BG;\, k) \to H_*(\mathcal{L}BG;\, k)$ and $\pi_* : H_*(\mathcal{L}BG;\, k) \to H_*(BG;\, k)$ be the induced map on homology, $\iota^* : H^*(\mathcal{L}BG;\, k) \to H^*(BG;\, k)$, $\pi^* : H^*(BG;\, k) \to H^*(\mathcal{L}BG;\, k)$ the induced maps on cohomology. Since $\pi \circ \iota = \mathbf{1}$ on $B_*(G)$, there are splittings of k-modules:

$$\begin{aligned}
H_*(\mathcal{L}BG;\, k) &\simeq H_*(BG;\, k) \oplus \mathrm{Ker}(\pi_*)\\
H^*(\mathcal{L}BG;\, k) &\simeq H^*(BG;\, k) \oplus \mathrm{Ker}(\iota^*).
\end{aligned}$$

By naturality, $\iota^*(\alpha \underset{S}{\cdot} \beta) = \iota^*(\alpha) \underset{S}{\cdot} \iota^*(\beta)$ and $\iota^*(\alpha \underset{1,\,S}{\cdot} \beta) = \iota^*(\alpha) \underset{1,\,S}{\cdot} \iota^*(\beta)$ for α, $\beta \in H^*(\mathcal{L}BG;\, k)$.

Lemma 2.1. *Let* $N^{\mathrm{cy}}_n(G,\, e) = \{(g_0,\, g_1,\, \dots\,,\, g_n) \in G^{n+1} \mid g_0 g_1 \dots g_n = e\}$ *for* $n = 0,\ 1,\ 2,\ \dots$ *. Then*

(i) $N^{\mathrm{cy}}_*(G,\, e)$ *is a subsimplicial set of* $N^{\mathrm{cy}}_*(G)$*.*

(ii) $Im(\iota) = N^{\mathrm{cy}}_*(G,\, e)$*.*

Proof. Part (i) follows since $N^{\mathrm{cy}}_*(G,\, e)$ is closed under the face maps and degeneracies of $N^{\mathrm{cy}}_*(G)$. For part (ii), let $(g_0,\, g_1,\, \dots\,,\, g_n) \in N^{\mathrm{cy}}_n(G,\, e)$. Then $g_0 g_1 \dots g_n = e$ and $g_0 = (g_1 g_2 \dots g_n)^{-1}$. Thus,

$$\iota(g_1,\, g_2,\, \dots\,, g_n) = (g_0,\, g_1,\, g_2,\, \dots\,, g_n).$$

$\square$

The group ring $k[G]$ is an algebra over the cyclic operad [10, 13.14.6], meaning that $k[G]$ supports a symmetric, bilinear inner product

$$\langle\ ,\ \rangle : k[G] \times k[G] \to k$$

satisfying $\langle ab,\, c\rangle = \langle a,\, bc\rangle$, for all $a,\, b,\, c \in k[G]$. By definition, for $g,\, h \in G$,

$$\langle g,\, h\rangle = \begin{cases} 1, & h = g^{-1}\\ 0, & h \neq g^{-1}. \end{cases}$$

Then $\langle\ ,\ \rangle$ is extended to be linear in each variable, resulting in a k-linear map on the tensor product: $\langle\ ,\ \rangle : k[G] \otimes k[G] \to k$. Since $\langle\ ,\ \rangle$ is symmetric, we also have $\langle a,\, bc\rangle = \langle ca,\, b\rangle$, i.e., $\langle\ ,\ \rangle$ is invariant under a cyclic shift of the product.

Lemma 2.2. *There is cochain map*

$$\Phi_n : \mathrm{Hom}_k(k[G]^{\otimes n},\ k[G]) \to \mathrm{Hom}_k(k[G]^{\otimes(n+1)},\ k), \quad n \geq 0,$$

given by

$$\Phi_n(f)(g_0,\, g_1,\, g_2,\, \dots\, ,\, g_n) = \langle g_0,\ f(g_1,\, g_2,\, \dots\, ,\, g_n)\rangle,$$

where $f : k[G]^{\otimes n} \to k[G]$ is a k-linear map and each $g_i \in G$.

Proof. For $f \in \mathrm{Hom}_k(k[G]^{\otimes(n-1)},\ k[G])$,

$$\Phi_n(\delta f)(g_0,\, g_1,\, \dots\, ,\, g_n) = \langle g_0,\ (\delta f)((g_1,\, g_2,\, \dots\, ,\, g_n)\rangle$$
$$= \langle g_0,\ g_1 f(g_2,\, \dots\, ,\, g_n)\rangle + \sum_{i=1}^{n-1}(-1)^i \langle g_0,\ f(g_1,\, \dots\, ,\, g_i g_{i+1},\, \dots\, ,\, g_n)\rangle$$
$$+ (-1)^n \langle g_0,\ f(g_1,\, g_2,\, \dots\, ,\, g_{n-1}) g_n\rangle.$$

On the other hand,

$$b^*(\Phi_{n-1}(f))(g_0,\, g_1,\, \dots\, ,\, g_n) = \Phi_{n-1}(f)(b(g_0,\, g_1,\, \dots\, ,\, g_n))$$
$$= \langle g_0 g_1,\ f(g_1,\, \dots\, ,\, g_n)\rangle + \sum_{i=1}^{n-1}(-1)^i \langle g_0,\ f(g_1,\, \dots\, ,\, g_i g_{i+1},\, \dots\, ,\, g_n)\rangle$$
$$+ (-1)^n \langle g_n g_0,\ f(g_1,\, g_2,\, \dots\, ,\, g_{n_1})\rangle.$$

Using the cyclic symmetries of the inner product $\langle\ ,\ \rangle$, we have

$$\Phi_n(\delta f) = b^*(\Phi_{n-1}(f)), \quad n \geq 1.$$

□

We adopt the following notation for elements of $\mathrm{Hom}_k(k[G]^{\otimes n},\, k[G])$ and $\mathrm{Hom}_k(k[G]^{\otimes(n+1)},\, k)$, recalling that $k[G]$ is a free k-module with basis given by the elements of G. For $g_0,\, g_1,\, \dots,\, g_n \in G$ and $h_1,\, h_2,\, \dots,\, h_n \in G$, let

$$(g_0,\, g_1,\, \dots\, ,\, g_n)^{\#} : k[G]^{\otimes n} \to k[G]$$

denote the k-linear map determined by

$$(g_0,\, g_1,\, \dots\, ,\, g_n)^{\#}(h_1,\, h_2,\, \dots\, ,\, h_n) = \begin{cases} g_0, & h_1 = g_1,\, \dots\, ,\, h_n = g_n, \\ 0 & \text{otherwise.} \end{cases}$$

Additionally, for $h_0 \in G$, let $(g_0,\, g_1,\, \dots\, ,\, g_n)^* : k[G]^{\otimes(n+1)} \to k$ be the k-linear map determined by

$$(g_0,\, g_1,\, \dots\, ,\, g_n)^*(h_0,\, h_1,\, \dots\, ,\, h_n) = \begin{cases} 1, & h_0 = g_0,\, h_1 = g_1,\, \dots\, ,\, h_n = g_n, \\ 0 & \text{otherwise.} \end{cases}$$

Under this notation,

$$\Phi_n\big((g_0,\, g_1,\, g_2,\, \dots\, ,\, g_n)^{\#}\big) = (g_0^{-1},\, g_1,\, g_2,\, \dots\, ,\, g_n)^*.$$

When G is finite, there is an inverse chain isomorphism

$$\Psi_n : \mathrm{Hom}_k(k[G]^{\otimes(n+1)},\, k) \to \mathrm{Hom}_k(k[G]^{\otimes n},\, k[G])$$

given by

$$\Psi_n\big((g_0,\, g_1,\, g_2,\, \dots\, ,\, g_n)^*\big) = (g_0^{-1},\, g_1,\, g_2,\, \dots,\, ,\, g_n)^{\#}.$$

For G infinite, let $HH^*_\rho(k[G],\, k[G])$ denote cohomology computed from cochains in $\mathrm{Hom}_k(k[G]^{\otimes *},\, k[G])$ that vanish on all but finitely many non-zero elements of $k[G]^{\otimes *}$. Also, let $HH^*_{\rho\mathcal{K}}(k[G])$ denote cohomology computed from cochains in $\mathrm{Hom}_k(k[G]^{\otimes(*+1)},\, k)$ that vanish on all but finitely many non-zero elements of $k[G]^{\otimes(*+1)}$. Then there are induced isomorphisms:

$$\Phi_* : HH^*_\rho(k[G],\, k[G]) \xrightarrow{\simeq} HH^*_{\rho\mathcal{K}}(k[G]),$$
$$\Psi_* : HH^*_{\rho\mathcal{K}}(k[G]) \xrightarrow{\simeq} HH^*_\rho(k[G],\, k[G]).$$

Definition 2.3. *For $\alpha_0,\, \alpha_1,\, \dots,\, \alpha_p \in G$ with*

$$\alpha = (\alpha_0,\, \alpha_1,\, \dots\, ,\, \alpha_p)^* \in \mathrm{Hom}_k(k[G]^{\otimes(p+1)},\, k),$$

we say that α is supported on $BG \simeq |N^{\mathrm{cy}}_*(G,\, e)|$ if $\alpha_0\alpha_1 \dots \alpha_p = e$.

Lemma 2.4. *If*

$$\alpha = (\alpha_0,\, \alpha_1,\, \dots,\, \alpha_p)^* \in \mathrm{Hom}_k(k[G]^{\otimes(p+1)},\, k) \quad \text{and}$$
$$\beta = (\beta_0,\, \beta_1,\, \dots,\, \beta_q)^* \in \mathrm{Hom}_k(k[G]^{\otimes(q+1)},\, k)$$

are supported on BG, then as cochains

$$\Psi_{p+q}(\alpha \underset{S}{\cdot} \beta) = \Psi_p(\alpha) \underset{G}{\cdot} \Psi_q(\beta).$$

Proof. Let $g_i \in G$, $i = 0,\, 1,\, 2,\, \dots,\, p+q$. Then

$$(\alpha \underset{S}{\cdot} \beta) \in \mathrm{Hom}_k(k[G]^{\otimes(p+q+1)},\, k)$$

is determined by

$$(\alpha \underset{S}{\cdot} \beta)(g_0,\, g_1,\, \dots,\, g_p,\, g_{p+1},\, \dots,\, g_{p+q}) =$$
$$\alpha((g_{p+1}g_{p+2}\dots g_{p+q}g_0),\ g_1,\ g_2,\ \dots,\ g_p)\beta((g_0g_1\dots g_p),\ g_{p+1},\ \dots,\ g_{p+q}).$$

Thus, $\alpha \underset{S}{\cdot} \beta$ is non-zero if and only if

$$(g_{p+1}g_{p+2}\dots g_{p+q}g_0) = \alpha_0, \quad g_1 = \alpha_1, \quad g_2 = \alpha_2,\ \dots,\ g_p = \alpha_p$$
$$(g_0g_1\dots g_{p-1}g_p) = \beta_0, \quad g_{p+1} = \beta_1, \quad g_{p+2} = \beta_2,\ \dots,\ g_{p+q} = \beta_q.$$

Hence,

$$g_0 = (g_{p+1}g_{p+2}\dots g_{p+q})^{-1}\alpha_0 = (\beta_1\beta_2\dots\beta_q)^{-1}\alpha_0 = \beta_0\alpha_0.$$

Now,

$$\begin{aligned}
&\Psi_{p+q}(\alpha \underset{S}{\cdot} \beta)\\
&= \Psi_{p+q}\big((\beta_0\alpha_0,\ \alpha_1,\ \alpha_2,\ \dots,\alpha_p,\ \beta_1,\ \beta_2,\ \dots,\ \beta_q)^*\big)\\
&= ((\beta_0\alpha_0)^{-1},\ \alpha_1,\ \alpha_2,\ \dots,\alpha_p,\ \beta_1,\ \beta_2,\ \dots,\ \beta_q)^\#\\
&= (\alpha_0^{-1}\beta_0^{-1},\ \alpha_1,\ \alpha_2,\ \dots,\alpha_p,\ \beta_1,\ \beta_2,\ \dots,\ \beta_q)^\#\\
&= (\alpha_0^{-1},\ \alpha_1,\ \dots,\ \alpha_p)^\# \underset{G}{\cdot} (\beta_0^{-1},\ \beta_1,\ \dots,\ \beta_q)^\#\\
&= \Psi_p(\alpha) \underset{G}{\cdot} \Psi_q(\beta).
\end{aligned}$$

$\square$

For $z \in G$, let $<z>$ denote the conjugacy class of z, and define

$$N_n^{\mathrm{cy}}(G, <z>) = \{(g_0, g_1, \ldots, g_n) \in G^{n+1} \mid g_0 g_1 \ldots g_n \in <z>\}.$$

Then $N_*^{\mathrm{cy}}(G, <z>)$ is a subsimplicial set of $N_*^{\mathrm{cy}}(G)$, and $HH_*(k[G])$ splits as a direct sum over the conjugacy classes of G.

Theorem 2.5. *[2] [9, 7.4.4] Let* Conj.(G) *denote the conjugacy classes of* G. *Then*

$$HH_*(k[G]) \simeq \sum_{<z> \in \mathrm{Conj.}(G)} HH_*(k[N_n^{\mathrm{cy}}(G, <z>)]).$$

Theorem 2.6. *For*

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_p)^* \in \mathrm{Hom}_k(k[G]^{\otimes(p+1)}, k) \quad \text{and}$$
$$\beta = (\beta_0, \beta_1, \ldots, \beta_q)^* \in \mathrm{Hom}_k(k[G]^{\otimes(q+1)}, k),$$

a necessary condition that $\alpha \underset{S}{\cdot} \beta \neq 0$ *is* $\alpha_0\alpha_1 \ldots \alpha_p = \beta_1\beta_2 \ldots \beta_q\beta_0$, *in which case*

$$\alpha \underset{S}{\cdot} \beta = (\beta_0(\alpha_1 \ldots \alpha_p)^{-1},\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p,\ \beta_1,\ \beta_2,\ \ldots,\ \beta_q)^*.$$

Proof. The proof follows from an argument similar to that of Lemma (2.4), beginning with a calculation of the form

$$(\alpha \underset{S}{\cdot} \beta)(g_0, g_1, \ldots, g_p, g_{p+1}, \ldots, g_{p+q}).$$

□

Corollary 2.7. *If* $\alpha_0\alpha_1 \ldots \alpha_p$ *and* $\beta_0\beta_1 \ldots \beta_q$ *are in different conjugacy classes of* G, *then* $\alpha \underset{S}{\cdot} \beta = 0$, *where* $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_p)^*$ *and* $\beta = (\beta_0, \beta_1, \ldots, \beta_q)^*$.

Proof. We cannot have $\alpha_0\alpha_1 \ldots \alpha_p = \beta_1\beta_2 \ldots \beta_q\beta_0$, since $\alpha_0\alpha_1 \ldots \alpha_p$ and $\beta_0\beta_1 \ldots \beta_q$ are in different conjugacy classes of G. □

For α and β as above, $\sigma \in G^{p+q}$, $j = 0, 1, \ldots, p-1$, let

$$(\alpha \underset{1,S}{\cdot} \beta)_j(\sigma) = \alpha(d_{j+1} \circ d_{j+2} \circ \ldots \circ d_{j+q-1})(\sigma)$$
$$\cdot\, \beta(d_0 \circ d_1 \circ \ldots \circ d_{j-1} \circ d_{j+q+1} \circ d_{j+q+2} \circ \ldots \circ d_{p+q-1})(\sigma)$$

be the jth summand in Steenrod's cup-one product.

Theorem 2.8. *Let*

$$\alpha = (\alpha_0,\, \alpha_1,\, \dots,\, \alpha_p)^* \in \mathrm{Hom}_k(k[G]^{\otimes(p+1)},\, k) \quad \text{and}$$
$$\beta = (\beta_0,\, \beta_1,\, \dots,\, \beta_q)^* \in \mathrm{Hom}_k(k[G]^{\otimes(q+1)},\, k)$$

be supported on BG. Then as cochains

$$\Psi_{p+q-1}((\alpha \underset{1,S}{\cdot} \beta)_j) = \Psi_p(\alpha) \underset{(j)}{\circ} \Psi_q(\beta),$$

i.e., over BG the jth term in Steenrod's cup-one product is the jth term in Gerstenhaber's pre-Lie system, after application of the cochain map Ψ_*.

Proof. Let $\sigma = (g_0,\, g_1,\, \dots,\, g_{p+q-1}) \in G^{p+q}$. Then

$$\begin{aligned}
&(\alpha \underset{1,S}{\cdot} \beta)_j(\sigma) = \\
&\alpha(g_0,\ g_1,\ \dots,\ g_j,\ (g_{j+1}g_{j+2}\cdots g_{j+q}),\ g_{j+q+1},\ \dots,\ g_{p+q-1}) \\
&\cdot \beta((g_{j+q+1}g_{j+q+2}\cdots g_{p+q-1}g_0g_1\cdots g_j),\ g_{j+1},\ g_{j+2},\ \dots\ g_{j+q})
\end{aligned}$$

Necessary conditions for $(\alpha \underset{1,S}{\cdot} \beta)_j(\sigma)$ to be non-zero are:

$$\begin{aligned}
&g_0 = \alpha_0,\ \ g_1 = \alpha_1,\ \dots,\ g_j = \alpha_j,\ \ (g_{j+1}g_{j+2}\cdots g_{j+q}) = \alpha_{j+1}, \\
&g_{j+q+1} = \alpha_{j+2},\ \ g_{j+q+2} = \alpha_{j+3},\ \dots,\ g_{p+q-1} = \alpha_p, \\
&(g_{j+q+1}g_{j+q+2}\cdots g_{p+q-1}g_0g_1\cdots g_j) = \beta_0, \\
&g_{j+1} = \beta_1,\ \ g_{j+2} = \beta_2,\ \dots,\ g_{j+q} = \beta_q.
\end{aligned}$$

Since $\alpha_0\alpha_1\dots\alpha_p = e$ and $\beta_0\beta_1\dots\beta_q = e$, we have $\alpha_{j+1} = \beta_0^{-1}$. Thus,

$$(\alpha \underset{1,s}{\cdot} \beta)_j = (\alpha_0,\ \alpha_1,\ \dots,\ \alpha_j,\ \beta_1,\ \beta_2,\ \dots,\ \beta_q,\ \alpha_{j+2},\ \alpha_{j+3},\ \dots,\ \alpha_p)^*,$$

provided $\alpha_{j+1} = \beta_0^{-1}$. Otherwise, $(\alpha \underset{1,s}{\cdot} \beta)_j$ is zero.

Let

$$\begin{aligned}
f &= \Psi_p(\alpha) = (\alpha_0^{-1},\ \alpha_1,\ \dots,\ \alpha_p)^{\#} \\
g &= \Psi_p(\beta) = (\beta_0^{-1},\ \beta_1,\ \dots,\ \beta_q)^{\#}
\end{aligned}$$

For $h_i \in G$, $i = 1, 2, 3, \dots, p+q-1$, we have

$$\begin{aligned}
&(f \underset{(j)}{\circ} g)(h_1,\, h_2,\, \dots,\, h_{p+q-1}) \\
&= f(h_1,\, h_2,\, \dots,\, h_j,\ g(h_{j+1},\, \dots,\, h_{j+q}),\ h_{j+q+1},\, \dots,\, h_{p+q-1}).
\end{aligned}$$

It follows that

$$f \underset{(j)}{\circ} g = (\alpha_0^{-1},\ \alpha_1,\ \ldots,\ \alpha_j,\ \beta_1,\ \beta_2,\ \ldots,\ \beta_q,\ \alpha_{j+2},\ \alpha_{j+3},\ \ldots,\ \alpha_p)^{\#}$$

under the condition that $\alpha_{j+1} = \beta_0^{-1}$. Thus,

$$\Psi_{p+q-1}((\alpha \underset{1,S}{\cdot} \beta)_j) = \Psi_p(\alpha) \underset{(j)}{\circ} \Psi_q(\beta).$$

□

Corollary 2.9. *Let*

$$\alpha = (\alpha_0,\, \alpha_1,\, \ldots,\, \alpha_p)^* \in \mathrm{Hom}_k(k[G]^{\otimes(p+1)},\, k) \quad \text{and}$$
$$\beta = (\beta_0,\, \beta_1,\, \ldots,\, \beta_q)^* \in \mathrm{Hom}_k(k[G]^{\otimes(q+1)},\, k)$$

be supported on BG. Then as cochains

$$\Psi_{p+q-1}(\alpha \underset{1,S}{\cdot} \beta) = \Psi_p(\alpha)\circ\Psi_q(\beta),$$

i.e., over BG Steenrod's cup-one product is Gerstenhaber's pre-Lie product, after application of the cochain map Ψ_*.

Proof. The proof follows from Theorem (2.8), the definition of Steenrod's cup-one, and the definition of the pre-Lie product. □

Steenrod's cup-i products measure the non-commutativity of the cup-$(i-1)$ products at the cochain level for any simplicial complex. Thus, the cup-2 product is a measure of the non-commutativity of Gerstenhaber's pre-Lie product for cochains supported on BG. For a combinatorial description of cup-i, see [11, 15]. The action of the mod 2 Steenrod algebra on $H^*(BG,\, \mathbf{Z}/2)$ can thus be seen in $HH^*(\mathbf{Z}/2[G],\, \mathbf{Z}/2[G])$ for cochains supported on BG. For $\alpha \in H^p(BG,\, \mathbf{Z}/2)$, $Sq^{p-1}(\alpha)$ is Gerstenhaber's pre-Lie product $\alpha \circ \alpha$ in cohomology and as cochains over $\mathbf{Z}/2$. Recall that $Sq^i(\alpha) = \alpha \underset{i-p,S}{\circ} \alpha$ is the cup $(i-p)$ product of α with itself, coefficients in $\mathbf{Z}/2$. See, for example, Mosher and Tangora [13].

3 Hochschild Hypercohomology

Let G_* be a simplicial group, such as the a model for the based loop space, ΩX, (of a connected CW-complex X). Then the face maps $d_i : G_p \to G_{p-1}$ and degeneracies $s_i : G_p \to G_{p+1}$, $i = 0, 1, \ldots, p$, are all group homomorphisms. Also, $(G^2)_* = (G \times G)_*$ becomes a simplicial group with $(G \times G)_p = G_p \times G_p$, face maps $d_i(g, h) = (d_i(g), d_i(h))$, and degeneracies $s_i(g, h) = (s_i(g), s_i(h))$. The construction can be iterated so that $(G^m)_*$ is a simplicial group for any positive integer m. By $HH_*(k[G_*]; k[G_*])$ is meant the homology of the (total complex) of the following bicomplex:

$$\begin{tikzcd}
\vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b"'] & \vdots \arrow[d, "b"'] & \\
k[G_0]^{\otimes 3} \arrow[d, "b"'] & k[G_1]^{\otimes 3} \arrow[l, "d"'] \arrow[d, "-b"'] & k[G_2]^{\otimes 3} \arrow[l, "d"'] \arrow[d, "b"'] & \cdots \arrow[l, "d"'] \\
k[G_0]^{\otimes 2} \arrow[d, "b"'] & k[G_1]^{\otimes 2} \arrow[l, "d"'] \arrow[d, "-b"'] & k[G_2]^{\otimes 2} \arrow[l, "d"'] \arrow[d, "b"'] & \cdots \arrow[l, "d"'] \\
k[G_0] & k[G_1] \arrow[l, "d"'] & k[G_2] \arrow[l, "d"'] & \cdots \arrow[l, "d"']
\end{tikzcd} \tag{3.1}$$

Specifically, $d : k[G_p]^{\otimes(n+1)} \to k[G_{p-1}]^{\otimes(n+1)}$ is given by $d = \sum_{i=0}^{p}(-1)^i d_i$, where now each $d_i : G_p^{n+1} \to G_{p-1}^{n+1}$ has form

$$d_i(g_0, g_1, \ldots, g_n) = (d_i(g_0), d_i(g_1), \ldots, d_i(g_n)).$$

Lemma 3.1. *The following diagram commutes:*

$$\begin{tikzcd}
k[G_{p-1}]^{\otimes(n+1)} \arrow[d, "b"'] & k[G_p]^{\otimes(n+1)} \arrow[l, "d"'] \arrow[d, "b"] \\
k[G_{p-1}]^{\otimes n} & k[G_p]^{\otimes n} \arrow[l, "d"']
\end{tikzcd}$$

Proof. The proof follows since $d_i(g_j g_{j+1}) = d_i(g_j)d_i(g_{j+1})$ for all $g_j, g_{j+1} \in G_p$, and b is given in terms of the product of elements in G_*. $\square$

Now define $d^{\mathrm{Tot}} : k[G_p]^{\otimes(n+1)} \to k[G_{p-1}]^{\otimes(n+1)} \oplus k[G_p]^{\otimes n}$ by

$$d^{\mathrm{Tot}}(z) = d(z) + (-1)^p b(z).$$

Corollary 3.2. *For the complex* (3.1), *we have* $d^{\mathrm{Tot}} \circ d^{\mathrm{Tot}} = 0$.

Proof. The proof follows from Lemma (3.1). $\square$

Thus, the total complex (3.1) is a chain complex. It follows from the work of Goodwillie [6] that if G_* is a simplicial group model for the based loop space ΩX, X a connected CW-complex, we have

$$HH_*(k[G_*],\, k[G_*]) \simeq H_*(\mathcal{L}X;\, k),$$

where $\mathcal{L}(X) = \mathrm{Maps}(S^1,\ X)$ is the free loop space and $H_*(\mathcal{L}X;\, k)$ denotes the singular homology of $\mathcal{L}X$.

Let $N_n^{\mathrm{cy}}(G_p) = G_p^{n+1}$. Then $N_*^{\mathrm{cy}}(G_*)$ is a bisimplicial object, for which the face maps

$$\begin{array}{ccc} G_{p-1}^{n+1} & \xleftarrow{d_i} & G_p^{n+1} \\ {\scriptstyle b_j}\big\downarrow & & \big\downarrow{\scriptstyle b_j} \\ G_{p-1}^{n} & \xleftarrow{d_i} & G_p^{n} \end{array}$$

form commutative squares. The notation $d_i^h = d_i : G_p^{n+1} \to G_{p-1}^{n+1}$, $i = 0, 1, 2, \ldots, p$, for the horizontal face maps and $d_j^v = b_j : G_p^{n+1} \to G_p^n$, $j = 0, 1, 2, \ldots, n$, for the vertical face maps is also used. Similarly there are commutative squares for the degeneracies

$$\begin{array}{ccc} G_p^{n+2} & \xrightarrow{s_i^h} & G_{p+1}^{n+2} \\ {\scriptstyle s_j^v}\big\uparrow & & \big\uparrow{\scriptstyle s_j^v} \\ G_p^{n+1} & \xrightarrow[s_i^h]{} & G_{p+1}^{n+1}, \end{array}$$

which uses the fact that $s_i^h : G_p \to G_{p+1}$ satisfies $s_i^h(e) = e$ (or $s_i^h(1) = 1$).

Let $B_*(G_*)$ be the bisimplicial object with $B_n(G_p) = G_p^n$ and face operators

$$\begin{array}{ccc} G_{p-1}^{n} & \xleftarrow{d_i^h} & G_p^{n} \\ {\scriptstyle d_j^v}\big\downarrow & & \big\downarrow{\scriptstyle d_j^v} \\ G_{p-1}^{n-1} & \xleftarrow{d_i^h} & G_p^{n-1}, \end{array} \tag{3.2}$$

where now $d_i^h(g_1,\, g_2,\, \ldots\, ,\, g_n) = (d_i(g_1),\, d_i(g_2),\, \ldots\, ,\, d_i(g_n))$ uses the simplicial structure of G_*, and

$$d_j^v(g_1,\, g_2,\, \ldots\, ,\, g_n) = \begin{cases} (g_2,\, g_3,\, \ldots\, ,\, g_n), & j = 0, \\ (g_1,\, \ldots\, , g_j g_{j+1},\, \ldots\, ,\, g_n), & 1 \le j \le n-1, \\ (g_1,\, g_2,\, \ldots\, ,\, g_{n-1}), & j = n, \end{cases}$$

uses the simplicial structure of the classifying space functor B_*. Then diagram (3.2) is commutative, and there is an analogous commutative square for the bisimplicial degeneracies of $B_*(G_*)$.

Lemma 3.3. *There are maps of bisimplicial objects*

$$\begin{aligned} &\iota : B_*(G_*) \to N_*^{\mathrm{cy}}(G_*), \quad \pi : N_*^{\mathrm{cy}}(G_*) \to B_*(G_*) \\ &\iota : G_p^n \to G_p^{n+1}, \quad \pi : G_p^{n+1} \to G_p^n \\ &\iota(g_1,\ g_2,\ \ldots\ ,\ g_n) = ((g_1 g_2 \ldots g_n)^{-1},\ g_1,\ g_2,\ \ldots\ ,\ g_n), \\ &\pi(g_0,\, g_1,\, g_2,\, \ldots\, ,\, g_n) = (g_1,\, g_2,\, \ldots\, ,\, g_n). \end{aligned}$$

Proof. It is already known that ι commutes with the simplicial structures arising from B_* and N_*^{cy}, while π commutes with those from N_*^{cy} to B_*. It is enough to show that the following diagrams commute for the face maps:

$$\begin{array}{ccc} B_n(G_p) & \xrightarrow{d_i^h} & B_n(G_{p-1}) \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota} \\ N_n^{\mathrm{cy}}(G_p) & \xrightarrow[d_i^h]{} & N_n^{\mathrm{cy}}(G_{p-1}) \end{array}$$

$$\begin{array}{ccc} N_n^{\mathrm{cy}}(G_p) & \xrightarrow[d_i^h]{} & N_n^{\mathrm{cy}}(G_{p-1}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B_n(G_p) & \xrightarrow{d_i^h} & B_n(G_{p-1}), \end{array}$$

and analogous diagrams commute for the degeneracies. $\square$

Let $||B_*(G_*)|| = |B_*(|G_*|)|$ be the double (or total) geometric realization of the bisimplicial set $B_*(G_*)$. Since $\pi \circ \iota = \mathbf{1}$ on $B_*(G_*)$, there are splittings of k-modules

$$\begin{aligned} H_*(||N_*^{\mathrm{cy}}(G_*)||;\, k) &\simeq H_*(||B_*(G_*)||;\, k) \oplus \mathrm{Ker}(\pi_*), \\ H^*(||N_*^{\mathrm{cy}}(G_*)||;\, k) &\simeq H^*(||B_*(G_*)||;\, k) \oplus \mathrm{Ker}(\iota^*). \end{aligned}$$

Recall [6] that when G_* is a simplicial group model for ΩX, X a connected CW-complex as above, we have $||B_*(G_*)|| \simeq X$. Thus,

$$H_*(\mathcal{L}X;\, k) \simeq H_*(X;\, k) \oplus \mathrm{Ker}(\pi_*).$$

Let $N_*^{\mathrm{cy}}(G_*,\, e)$ be defined by

$$N_n^{\mathrm{cy}}(G_p,\, e) = \{(g_0,\, g_1,\, \ldots\, ,\, g_n) \in G_p^{n+1} \mid g_0 g_1 \ldots g_n = e\}.$$

Then $N_*^{\mathrm{cy}}(G_*,\, e)$ is a sub-bisimplicial set of $N_*^{\mathrm{cy}}(G_*)$ and $\mathrm{Im}(\iota) = N_*^{\mathrm{cy}}(G_*,\, e)$. Thus,

$$||N_*^{\mathrm{cy}}(G_*,\, e)|| = ||B_*(G_*)||.$$

Let $HH_{\mathcal{K}}^*(k[G_*])$ denote the homology of the $\mathrm{Hom}_k(\ ,k)$-dual of the bicomplex (3.1), i.e.,

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\longrightarrow & & \mathrm{Hom}_k(k[G_p]^{\otimes(n+2)},\, k) & \xrightarrow{d^*} & \mathrm{Hom}_k(k[G_{p+1}]^{\otimes(n+2)},\, k) & \longrightarrow & \ldots \\
 & & {\scriptstyle (-1)^p b^*}\uparrow & & \uparrow{\scriptstyle (-1)^{(p+1)} b^*} & & \\
\longrightarrow & & \mathrm{Hom}_k(k[G_p]^{\otimes(n+1)},\, k) & \xrightarrow{d^*} & \mathrm{Hom}_k(k[G_{p+1}]^{\otimes(n+1)},\, k) & \longrightarrow & \ldots \\
 & & \uparrow & & \uparrow & &
\end{array} \tag{3.3}$$

We refer to the above complex (3.3) as the $(b^*,\, d^*)$-bicomplex.

For any bisimplicial k-module $X_{p,\,n}$ that is free over k, let p denote the horizontal direction and n the vertical direction. For $\varphi \in \mathrm{Hom}_k(X_{p,\,n},\, k)$, let

$$d^{\mathrm{Tot}}(\varphi) = (d^h)^*(\varphi) + (-1)^p (d^v)^*(\varphi) \in \mathrm{Hom}_k(X_{p+1,\,n},\, k) \oplus \mathrm{Hom}_k(X_{p,\,n+1},\, k),$$

where $d^h = \sum_{i=0}^{p+1}(-1)^i d_i^h$, $d^v = \sum_{i=0}^{n+1}(-1)^i d_i^v$. Let $\alpha \in \mathrm{Hom}_k(X_{p,\,\ell},\, k)$ and $\beta \in \mathrm{Hom}_k(X_{q,\,m},\, k)$. For $\sigma \in X_{p+q,\,\ell+m}$, define the front $(p,\, \ell)$-face of σ, $f_{p,\,\ell}(\sigma)$, as

$$f_{p,\,\ell}(\sigma) = d_{p+1}^h \circ d_{p+2}^h \circ \ldots \circ d_{p+q}^h \circ d_{\ell+1}^v \circ \ldots \circ d_{\ell+m}^v(\sigma).$$

Define the back $(q,\, m)$-face of σ, ${}_{(q,\,m)}f(\sigma)$, as

$$({}_{(q,\,m)}f)(\sigma) = d_0^h \circ h_1^h \circ \ldots \circ d_{p-1}^h \circ d_0^v \circ d_1^v \circ \ldots \circ d_{\ell-1}^v(\sigma) = (h_0^h)^p \circ (d_0^v)^\ell(\sigma).$$

Define the bisimplicial cup product $\alpha \underset{S}{\cdot} \beta \in \mathrm{Hom}_k(X_{p+q,\,\ell+m},\, k)$ by

$$(\alpha \underset{S}{\cdot} \beta)(\sigma) = (-1)^{\ell q} \alpha(f_{p,\,\ell}(\sigma)) \cdot \beta({}_{(q,\,m)}f(\sigma)).$$

Lemma 3.4. *For $\alpha \in \mathrm{Hom}_k(X_{p,\,\ell},\, k)$ and $\beta \in \mathrm{Hom}_k(X_{q,\,m},\, k)$, we have:*

$$d^{\mathrm{Tot}}(\alpha \underset{S}{\cdot} \beta) = d^{\mathrm{Tot}}(\alpha) \underset{S}{\cdot} \beta + (-1)^{p+\ell} \alpha \underset{S}{\cdot} d^{\mathrm{Tot}}(\beta).$$

Proof.

$$\begin{aligned}
&d^{\mathrm{Tot}}(\alpha \underset{S}{\cdot} \beta) = (d^h)^*(\alpha \underset{S}{\cdot} \beta) + (-1)^{p+q}(d^v)^*(\alpha \underset{S}{\cdot} \beta) \\
&= (d^h)^*(\alpha) \underset{S}{\cdot} \beta + (-1)^{p+\ell} \alpha \underset{S}{\cdot} (d^h)^*(\beta) \\
&+ (-1)^p (d^v)^*(\alpha) \underset{S}{\cdot} \beta + (-1)^{p+q+\ell} \alpha \underset{S}{\cdot} (d^v)^*(\beta) \\
&= d^{\mathrm{Tot}}(\alpha) \underset{S}{\cdot} \beta + (-1)^{p+\ell} \alpha \underset{S}{\cdot} d^{\mathrm{Tot}}(\beta).
\end{aligned}$$

∎

Thus, for $\alpha \in HH^{p+\ell}_{\mathcal{K}}(k[G_*])$ and $\beta \in HH^{q+m}_{\mathcal{K}}(k[G_*])$, we have

$$\alpha \underset{S}{\cdot} \beta \in HH^{p+q+\ell+m}_{\mathcal{K}}(k[G_*]),$$

which realizes the topological cup product on $||N^{\mathrm{cy}}_*(G_*)||$.

Steenrod's cup-one can be defined on a bisimplicial cochain complex by adding, with appropriate signs, the simplicial cup-one constructions in the horizontal and vertical directions. Let $\alpha^{p,\,\ell} \in \mathrm{Hom}_k(X_{p,\,\ell},\, k)$ and $\beta^{q,\,m} \in \mathrm{Hom}_k(X_{q,\,m},\, k)$. First define

$$\alpha^{p,\,\ell} \underset{1,\,h}{\cdot} \beta^{q,\,m} \in \mathrm{Hom}_k(X_{p+q-1,\,m+\ell},\, k)$$

by

$$\begin{aligned}
&\alpha^{p,\,\ell} \underset{1,\,h}{\cdot} \beta^{q,\,m} = \\
&(-1)^{\ell(q-1)} \sum_{j=0}^{p-1} (-1)^{(p-1-j)(q-1)} \alpha^{p,\,\ell}(d^h_{j+1} \circ d^h_{j+2} \circ \ldots \circ d^h_{j+q-1} \circ (d^v_{\ell+1})^{[m]}) \cdot \\
&\qquad \beta^{q,\,m}(d^h_0 \circ d^h_1 \circ \ldots \circ d^h_{j-1} \circ d^h_{j+q+1} \circ \ldots \circ d^h_{p+q-1} \circ (d^v_0)^{\ell}),
\end{aligned}$$

where $(d^v_{\ell+1})^{[m]} = d^v_{\ell+1} \circ d^v_{\ell+2} \circ \ldots \circ d^v_{\ell+m}$. Then define

$$\alpha^{p,\,\ell} \underset{1,\,v}{\cdot} \beta^{q,\,m} \in \mathrm{Hom}_k(X_{p+q,\,m+\ell-1},\ k)$$

by

$$\alpha^{p,\,\ell} \underset{1,\,v}{\cdot} \beta^{q,\,m} =$$

$$(-1)^{pq+q(\ell-1)} \sum_{j=0}^{\ell-1} (-1)^{(\ell-1-j)(m-1)} \alpha((d^h_0)^q \circ d^v_{j+1} \circ d^v_{j+2} \circ \ldots \circ d^v_{j+m-1}) \cdot$$
$$\beta((d^h_{q+1})^{[p]} \circ d^v_0 \circ d^v_1 \circ \ldots \circ d^v_{j-1} \circ d^v_{j+m+1} \circ \ldots \circ d^v_{\ell+m-1}).$$

Set $\alpha^{p,\,\ell} \underset{1,\,S}{\cdot} \beta^{q,\,m} = \alpha^{p,\,\ell} \underset{1,\,h}{\cdot} \beta^{q,\,m} + \alpha^{p,\,\ell} \underset{1,\,v}{\cdot} \beta^{q,\,m}$.

Lemma 3.5. *For $\alpha^{p,\ell} \in \mathrm{Hom}_k(X_{p,\ell}, k)$ and $\beta^{q,\,m} \in \mathrm{Hom}_k(X_{q,\,m}, k)$, we have*

$$d^{\mathrm{Tot}}(\alpha^{p,\,\ell} \underset{1,\,S}{\cdot} \beta^{q,\,m}) = d^{\mathrm{Tot}}(\alpha^{p,\,\ell}) \underset{1,\,S}{\cdot} \beta^{q,\,m} + (-1)^{p+\ell-1} \alpha^{p,\,\ell} \underset{1,\,S}{\cdot} d^{\mathrm{Tot}}(\beta^{q,\,m})$$
$$+ (-1)^{\ell+p}[\alpha^{p,\,\ell} \underset{S}{\cdot} \beta^{q,\,m} - (-1)^{(p+\ell)(q+m)} \beta^{q,\,m} \underset{S}{\cdot} \alpha^{p,\,\ell}].$$

Proof.

$$
\begin{aligned}
&d^{\mathrm{Tot}}(\alpha^{p,\,\ell} \underset{1,\,S}{\cdot} \beta^{q,\,m}) = \\
&(d^h)^*(\alpha) \underset{1,\,h}{\cdot} \beta + (-1)^{p+\ell-1}\alpha \underset{1,\,h}{\cdot} (d^h)^*(\beta) \\
&+ (-1)^p (d^v)^*(\alpha) \underset{1,\,v}{\cdot} \beta + (-1)^{p+q+\ell-1}\alpha \underset{1,\,v}{\cdot} (d^v)^*(\beta) \\
&+ (-1)^{p+\ell}\alpha \underset{S}{\cdot} \beta - (-1)^{(p+\ell)(q+m)+\ell+p}\beta \underset{S}{\cdot} \alpha \\
&+ (d^h)^*(\alpha \underset{1,\,v}{\cdot} \beta) + (-1)^{p+q-1}(d^v)^*(\alpha \underset{1,\,h}{\cdot} \beta) \\
&= (d^h)^*(\alpha) \underset{1,\,h}{\cdot} \beta + (d^h)^*(\alpha) \underset{1,\,v}{\cdot} \beta \\
&+ (-1)^p[(d^v)^*(\alpha) \underset{1,\,h}{\cdot} \beta + (d^v)^*(\alpha) \underset{1,\,v}{\cdot} \beta] \\
&+ (-1)^{p+\ell-1}[\alpha \underset{1,\,h}{\cdot} (d^h)^*(\beta) + \alpha \underset{1,\,v}{\cdot} (d^h)^*(\beta)] \\
&+ (-1)^{p+q+\ell-1}[\alpha \underset{1,\,h}{\cdot} (d^v)^*(\beta) + \alpha \underset{1,\,v}{\cdot} (d^v)^*(\beta)] \\
&+ (-1)^{\ell+p}[\alpha \underset{S}{\cdot} \beta - (-1)^{(p+\ell)(q+m)}\beta \underset{S}{\cdot} \alpha] \\
&= d^{\mathrm{Tot}}(\alpha) \underset{1,\,S}{\cdot} \beta + (-1)^{p+\ell-1}\alpha \underset{1,\,S}{\cdot} d^{\mathrm{Tot}}(\beta) \\
&+ (-1)^{\ell+p}[\alpha \underset{S}{\cdot} \beta - (-1)^{(p+\ell)(q+m)}\beta \underset{S}{\cdot} \alpha].
\end{aligned}
$$

□

Thus $\alpha \underset{1,\,S}{\cdot} \beta$ is a construction of the cup-one product on cochains of a bisimplicial complex. By the theory of acyclic carriers [14, p. 25–27], the cup-i products exist on the cochains of a bisimplicial complex as well, and are defined in terms of the bisimplicial face maps.

Theorem 3.6. *As modules over the mod 2 Steenrod algebra,*

$$H^*(||N_*^{\mathrm{cy}}(G_*)||;\, \mathbf{Z}/2) \simeq H^*(||B_*(G_*)||;\, \mathbf{Z}/2) \oplus \mathrm{Ker}(\iota^*),$$

where ι is defined in Lemma (3.3)*.*

Proof. This follows since

$$\iota : B_*(G_*) \to N_*^{\mathrm{cy}}(G_*) \quad \text{and} \quad \pi : N_*^{\mathrm{cy}}(G_*) \to B_*(G_*)$$

are maps of bisimplicial objects, $\pi \circ \iota = \mathbf{1}$ on $B_*(G_*)$, and the construction of the cup-i products involves only the bisimplicial structure of these objects. □

4 A Hybrid Complex

Let G_* be a simplicial group, each G_n finite. For G_n infinite, use the completion of the group ring $\hat{k}[G]$. We define a cochain bicomplex $\mathrm{Hom}_k(k[G_*]^{\otimes *},\, k[G_*])$ with columns $(\mathrm{Hom}_k(k[G_p]^{\otimes *},\, k[G_p]),\, \delta)$ and rows $(\mathrm{Hom}_k(k[G_*]^{\otimes n},\, k[G_*]),\, D)$, where D is a certain sum of inverse images of the individual face operators $d_i : G_{p+1} \to G_p$. Define

$$D : \mathrm{Hom}_k(k[G_p]^{\otimes n},\, k[G_p]) \to \mathrm{Hom}_k(k[G_{p+1}]^{\otimes n},\, k[G_{p+1}])$$

as follows. For $f \in \mathrm{Hom}_k(k[G_p]^{\otimes n},\, k[G_p])$ and $(g_1,\, g_2,\, \dots,\, g_n) \in G_{p+1}^n \hookrightarrow k[G_{p+1}]^{\otimes n}$, set

$$(Df)(g_1,\, g_2,\, \dots,\, g_n) = \sum_{i=0}^{p+1} (-1)^i C_i\big(f(d_i(g_1),\, d_i(g_2),\, \dots,\, d_i(g_n))\big).$$

For $f(d_i(g_1),\, d_i(g_2),\, \dots,\, d_i(g_n)) = \gamma \in G_p$, let $C_i(\gamma) = \sum_{h_i \in d_i^{-1}(\gamma)} \in k[G_{p+1}]$. Then extend $C_i : k[G_p] \to k[G_{p+1}]$ to be k-linear for $f(d_i(g_1),\, \dots,\, d_i(g_n)) = \sum_{j=1}^{m} k_j \gamma_j \in k[G_p]$. Using the identities $d_j d_j = d_{j-1} d_i$, $i < j$, it follows that $D \circ D = 0$. Define the $(\delta,\, D)$-bicomplex as

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\delta\uparrow & & -\delta\uparrow & & \\
\mathrm{Hom}_k(k[G_0]^{\otimes 2},\, k[G_0]) & \xrightarrow{D} & \mathrm{Hom}_k(k[G_1]^{\otimes 2},\, k[G_1]) & \xrightarrow{D} & \dots \\
\delta\uparrow & & -\delta\uparrow & & \\
\mathrm{Hom}_k(k[G_0],\, k[G_0]) & \xrightarrow{D} & \mathrm{Hom}_k(k[G_1],\, k[G_1]) & \xrightarrow{D} & \dots \\
\delta\uparrow & & -\delta\uparrow & & \\
\mathrm{Hom}_k(k,\, k[G_0]) & \xrightarrow{D} & \mathrm{Hom}_k(k,\, k[G_1]) & \xrightarrow{D} & \dots
\end{array}$$

Lemma 4.1. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Hom}_k(k[G_p]^{\otimes(n+1)},\, k[G_p]) & \xrightarrow{D} & \mathrm{Hom}_k(k[G_{p+1}]^{\otimes(n+1)},\, k[G_{p+1}]) \\
\delta\uparrow & & \delta\uparrow \\
\mathrm{Hom}_k(k[G_p]^{\otimes n},\, k[G_p]) & \xrightarrow{D} & \mathrm{Hom}_k(k[G_{p+1}]^{\otimes n},\, k[G_{p+1}]).
\end{array}$$

Proof. For $f \in \mathrm{Hom}_k(k[G_p]^{\otimes n},\, k[G_p])$, we have

$$\begin{aligned}
&\delta(Df)(g_1,\, g_2,\, \dots,\, g_{n+1})\\
&g_1 \cdot (Df)(g_2,\, g_3,\, \dots,\, g_{n+1}) - (Df)(g_1 \cdot g_2,\, g_3,\, \dots,\, g_{n+1}) + \cdots\\
&(-1)^n (Df)(g_1,\, g_2,\, \dots,\, g_n \cdot g_{n+1}) + (-1)^{n+1}(Df)(g_1,\, g_2,\, \dots,\, g_n) \cdot g_{n+1}.
\end{aligned}$$

Now, each Df can be expanded as a sum $\sum_{i=0}^{p+1}(-1)^i C_i$. On the other hand,

$$\begin{aligned}
&D(\delta f)(g_1,\, g_2,\, ,\dots,\, g_{n+1})\\
&= \sum_{i=1}^{p+1} C_i(\delta f)(d_i(g_1),\, d_i(g_2),\, \dots,\, d_i(g_{n+1})).
\end{aligned}$$

Note that for $g \in G_p$, $d_i^{-1}(g)$ is a coset in G_{p+1} of the normal subgroup $\mathrm{Ker}(d_i) \lhd G_{p+1}$. Thus, we have, for example,

$$g_1 \cdot C_0(f(d_0(g_2),\, \dots,\, d_0(g_{p+1}))) = C_0(d_0(g_1) \cdot f(d_0(g_2),\, \dots,\, d_0(g_{p+1}))),$$

and it follows that $\delta \circ D = D \circ \delta$. $\square$

Lemma 4.2. *There is a cochain map from the* $(\delta,\, D)$*-bicomplex to the* $(b^*,\, d^*)$*-bicomplex in* (3.3)

$$\Phi_n : \mathrm{Hom}_k(k[G_p]^{\otimes n},\, k[G_p]) \to \mathrm{Hom}_k(k[G_p]^{\otimes (n+1)},\, k)$$

given by

$$\Phi_n(f)(g_0,\, g_1,\, g_2,\, \dots,\, g_n) = \langle g_0,\ f(g_1,\, g_2,\, \dots,\, g_n)\rangle,$$

where $\langle\ \rangle : k[G_p] \otimes k[G_p] \to k$ *is defined just prior to Lemma* (2.2)*.*

Proof. From Lemma (2.2), $\Phi_n \circ \delta = b^* \circ \Phi_{n-1}$. It remains to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_k(k[G_p]^{\otimes n},\, k[G_p]) & \xrightarrow{D} & \mathrm{Hom}_k(k[G_{p+1}]^{\otimes n},\, k[G_{p+1}])\\
{\scriptstyle \Phi_n}\big\downarrow & & {\scriptstyle \Phi_n}\big\downarrow\\
\mathrm{Hom}_k(k[G_p]^{\otimes (n+1)},\, k) & \xrightarrow{d^*} & \mathrm{Hom}_k(k[G_{p+1}]^{\otimes (n+1)},\, k).
\end{array}$$

For $g_i \in G_{p+1}$, $f \in \mathrm{Hom}_k(k[G_p]^{\otimes n},\, k[G_p])$,

$$d^*(\Phi_n(f))(g_0,\, g_1,\, \dots,\, g_n) = \sum_{i=0}^{p+1}(-1)^i \langle d_i(g_0),\ f(d_i(g_1),\, \dots,\, d_i(g_n))\rangle.$$

On the other hand,

$$\Phi_n(Df)(g_0,\, g_1,\, \dots\,,\, g_n) = \sum_{i=0}^{p+1} \langle g_0,\ C_i(f(d_i(g_1),\, \dots\,,\, d_i(g_n)))\rangle.$$

Consider the case in which $z_0 = f(d_i(g_1),\, \dots\,,\, d_i(g_n)) \in G_p$. Then

$$\langle d_i(g_0),\, z_0\rangle = 1 \Leftrightarrow d_i(g_0)^{-1} = z_0 \Leftrightarrow d_i(g_0^{-1}) = z_0.$$

Also, $\langle g_0,\, C_i(z_0)\rangle = 1 \Leftrightarrow g_0^{-1} \in d_i^{-1}(z_0)$. Thus, $\langle d_i(g_0),\, z_0\rangle = \langle g_0,\, C_i(z_0)\rangle$. By linearity, $d^* \circ \Phi_n = \Phi_n \circ D$. □

In the case when each G_p is finite, Φ is an isomorphism of cochain complexes, with inverse

$$\Psi_n((g_0,\, g_1,\, \dots\,,\, g_n)^*) = (g_0^{-1},\, g_1,\, \dots\,,\, g_n)^{\#}.$$

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